Title

Joe Puccio

February 2, 2022

Proof. O is open $\rightarrow F = O^{\complement}$ is closed.

$F = O^{\complement}$ is closed iff it contains all its limit points.
Take $x =$ limit point of $O^{\complement}$ Goal: show that $x \in O^{\complement}$

$\forall \epsilon > 0$ $V_{\epsilon}(x) \cap O^{\complement}$ contains at least a point in $y\ in\ O^{\complement}$ other than x.

Assume that $x \in O$ $\exists V_y(x) \subseteq O$

$V_y(x) \cap O^{\complement} = \emptyset$

In particular for $\epsilon = y$

We should have
$V_y(x) \cap O^{\complement}$ contains at least a point y in $O^{\complement}$ i.e. $V_y(x) \cap O^{\complement} \neq \emptyset$

Conclusion: $x \notin O \rightleftarrows x \in O^{\complement} \rightarrow$ $O^{\complement}$ contains all its limit points $\rightarrow$O$^{\complement}$ is closed.

Reverse Implication:

We assume that $O^{\complement}$ is closed. We want to show that O is open.

Take $x \in O$

x is not a limit point of $O^{\complement}$
$\exists V_{\epsilon}(x) | V_{\epsilon}(x) \cap O^{\complement} = \{\emptyset\ or \{x\}\}$

$V_{\epsilon}(x) \cap O^{\complement}$ can not be equal to x because $x \in O$
So $V_{\epsilon}(x) \cap O^{\complement} = \emptyset \rightarrow V_{\epsilon}(x) \subset \rightarrow O$ open.

□

Compact Sets

Definition $K \neq \emptyset$ $K \subset \mathbb{R}$
K is compact if every sequence $(x_n) \in K$ has a convergence subsequence $\in K$.
$K \subset \mathbb{R}$ i.e. $\forall (x_n) \in K$ $\exists \bar{x}$ and $\exists (x_{n_k})|$ $lim(x_{n_k}) = \bar{x}$

$\bar{x} \in K$

Theorem (HB - HEINE-BOREL)

K is compact iff K is closed and bounded.

Proof. K compact $\implies$ K closed and bounded

1) K bounded $\exists M > 0 \forall x \in K \mid |x| \leq M$
Assume this is not true $\implies$ $\forall M > 0$ $\exists x \in K|$ $|x| > M$
$M = 1 \implies \exists x_1 \in K \mid |x_1| > 1$

Missing part.

x_n is not bounded.

But $x_n \in K$ and K compact. $\implies$ $\exists \bar{x} \in K$ and $(x_{n_k}$ subsequence of (x_n) $|(x_{n_k} \to \bar{x} \implies |(x_{n_k}| \to |\bar{x}|$

$|(x_{n_k}| > n_k$

$n_k = m \quad |x_m| > m$

(R Triangle Inequality) $\implies$ $(|(x_{n_k}|)$ being convergent is bounded contradiction with $|(x_{n_k}| > n_k$ which implies $lim|(x_{n_k}| = \infty$

Conclusion

K compact $\implies$ K bounded.

K compact $\implies$ K closed?

Take $x =$ limit point of K

K compact $\implies$ $\exists \bar{x} \in K, \exists (x_{n_k})|(x_{n_k})(x_{n_k} \to x)$

because $x_n \to x$

$(x_{n_k}$ subsequence $\implies (x_{n_k} \to x$

$\exists (x_n)\ x_n \in K : lim(n \to \infty) of x_n = x$

$x_n \neq x\ \forall n$

Take $(x_n) \in K \implies x_n$ is bounded because K is bounded

x_n bounded $\implies x_n$ has a convergent subsequence (x_{n_k})

i.e. $\exists \bar{x} : lim(k) x_{n_k} = \bar{x}\ \bar{x} \neq x_{n_k}$

It remains to show that $\bar{x} \in K$

2 cases
1) $\bar{x} =$ limit point of K.
i.e. $x_{n_k} \neq \bar{x} \forall k$
then $\bar{x} \in K$ because K closed.

(it contains all its limit points)

2) $\exists k_0\ |x_{n_{k_0}} = \bar{x}$ then $\bar{x} \in K$

Open Cover

Definition: An open cover of A

O_i open sets $i \in I$

$A \subset \cup_{i \in I}\ O_i$p

□